

NOTES ON THETA SERIES FOR NIEMEIER LATTICES

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ABSTRACT. Some explicit expressions are given for the theta series of Niemeier lattices. As an application, we present some of their congruence relations.

1. INTRODUCTION

The main object of this note is the theta series $\vartheta_{\mathcal{L}}^{(n)}$ associated with the Niemeier lattice \mathcal{L} :

$$\vartheta_{\mathcal{L}}^{(n)} = \vartheta_S^{(n)}(Z) := \sum_{X \in M_{24,n}(\mathbb{Z})} \exp(\pi \sqrt{-1} \operatorname{tr}(S[X]Z)), \quad Z \in \mathbb{H}_n,$$

where $S \in 2\operatorname{Sym}_{24}^*(\mathbb{Z})$ is the Gram matrix of \mathcal{L} , $S[X] := {}^tX S X$, and \mathbb{H}_n is the Siegel upper half-space of degree n . In the following, we will sometimes refer to $\vartheta_{\mathcal{L}}^{(n)}$ as the Niemeier theta series.

A Niemeier lattice \mathcal{L} is one of the 24 positive definite, even, unimodular lattices of rank 24. Therefore, the Niemeier theta series $\vartheta_{\mathcal{L}}^{(n)}$ becomes a Siegel modular form of weight 12 and degree n . In this note, we give explicit expressions for $\vartheta_{\mathcal{L}}^{(n)}$, using some modular forms of weight 12 with integral Fourier coefficients for the cases $n = 2$ and 3.

Theorem 1. *Let \mathcal{L} be a Niemeier lattice with Coxeter number $h = h_{\mathcal{L}}$. The theta series $\vartheta_{\mathcal{L}}^{(3)}$ has the following expression:*

$$(1) \quad \begin{aligned} \vartheta_{\mathcal{L}}^{(3)} = & (E_4^{(3)})^3 + (24h - 720)Y_{12}^{(3)} + (48h^2 - 2880h + 43200)X_{12}^{(3)} \\ & + (48h^3 - 288h^2 + 3144h - 1131120)F_{12}, \end{aligned}$$

where $E_4^{(3)}$ is the Eisenstein series of weight 4 and degree 3; $Y_{12}^{(3)}$, $X_{12}^{(3)}$ are Siegel modular forms of weight 12 with integral Fourier coefficients defined in Section 4.1; and F_{12} is Miyawaki's cusp form of weight 12 (cf. [7]).

The expression (1) leads to congruence relations among the Niemeier theta series. For example, the congruence relations between the Coxeter numbers are related to those of the Niemeier theta series.

Corollary 1.1. *Let \mathcal{L}_i ($i = 1, 2$) be Niemeier lattices with Coxeter number $h_i := h_{\mathcal{L}_i}$. If $h_1 \equiv h_2 \pmod{m}$ for an integer m , then*

$$\vartheta_{\mathcal{L}_1}^{(3)} \equiv \vartheta_{\mathcal{L}_2}^{(3)} \pmod{m}.$$

Conway and Sloane listed such lattices according to the glue code and named them $\alpha, \beta, \dots, \omega$ (cf. Conway and Sloane [4], p. 407, Table 16.1).

Theorem 2. *Let α , ω , δ , and ψ be some of the Niemeier lattices defined above. The following congruence relations hold:*

$$\begin{aligned}\vartheta_\alpha^{(3)} &\equiv \vartheta_\omega^{(3)} \equiv \vartheta_{[4,2,6]}^{(3)} \pmod{23}, \\ \vartheta_\delta^{(3)} &\equiv \vartheta_\psi^{(3)} \equiv \vartheta_{[2,2,12]}^{(3)} \pmod{23},\end{aligned}$$

where we use the following abbreviations: $[4, 2, 6] = \begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix}$, and $[2, 2, 12] = \begin{pmatrix} 2 & 1 \\ 1 & 12 \end{pmatrix}$ (see (10) in Section 4.2).

These congruence relations lead to the following fact (also cf. [10]).

Corollary 2.1. (1) *We have*

$$\Theta(\vartheta_\alpha^{(2)}) \equiv \Theta(\vartheta_\omega^{(2)}) \equiv \Theta(\vartheta_\delta^{(2)}) \equiv \Theta(\vartheta_\psi^{(2)}) \equiv 0 \pmod{23},$$

where Θ is the theta operator (the generalized Ramanujan operator) defined in Section 2.4.

(2) *The Siegel modular forms*

$$\vartheta_\alpha^{(3)}, \quad \vartheta_\omega^{(3)}, \quad \vartheta_\delta^{(3)}, \quad \vartheta_\psi^{(3)}$$

are the mod 23 singular modular forms with the maximal 23-rank 2 in the sense of Section 2.4 (cf. [1]).

2. PRELIMINARIES

2.1. Notation. We begin by stating the notation that we will use. Let $\Gamma_n := Sp_n(\mathbb{Z})$ be the Siegel modular group of degree n , and let \mathbb{H}_n be the Siegel upper half-space of degree n . We denote by $M_k(\Gamma_n)$ the \mathbb{C} -vector space of all Siegel modular forms of weight k for Γ_n , and denote by $S_k(\Gamma_n)$ the subspace of cusp forms.

Any $F(Z)$ in $M_k(\Gamma_n)$ has a Fourier expansion of the form

$$F(Z) = \sum_{0 \leq T \in \text{Sym}_n^*(\mathbb{Z})} a(F; T) q^T, \quad q^T := \exp(2\pi\sqrt{-1}\text{tr}(TZ)), \quad Z \in \mathbb{H}_n,$$

where

$$\text{Sym}_n^*(\mathbb{Z}) := \{ T = (t_{ij}) \in \text{Sym}_n(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z} \}.$$

We will write the Fourier coefficient corresponding to $T \in \text{Sym}_n^*(\mathbb{Z})$ as $a(F; T)$.

For a subring R of \mathbb{C} , let $M_k(\Gamma_n)_R \subset M_k(\Gamma_n)$ denote the R -module of all modular forms whose Fourier coefficients lie in R .

2.2. Formal q -expansion. For $T = (t_{ij}) \in \text{Sym}_n^*(\mathbb{Z})$ and $Z = (z_{ij}) \in \mathbb{H}_n$, we define $q_{ij} := \exp(2\pi\sqrt{-1}z_{ij})$. Then,

$$q^T := \exp(2\pi\sqrt{-1}\text{tr}(TZ)) = \prod_{i < j} q_{ij}^{2t_{ij}} \prod_{i=1}^n q_{ii}^{t_{ii}}.$$

Therefore, we may consider $F \in M_k(\Gamma_n)_R$ as an element of the formal power series ring

$$F = \sum a(F; T) q^T \in R[q_{ij}, q_{ij}^{-1}][q_{11}, \dots, q_{nn}].$$

For a prime number p , we denote by $\mathbb{Z}_{(p)}$ the local ring of p -integral rational numbers. For two elements

$$F_i = \sum a(F_i; T) q^T \in \mathbb{Z}_{(p)}[q_{ij}, q_{ij}^{-1}] \llbracket q_{11}, \dots, q_{nn} \rrbracket \quad (i = 1, 2),$$

we write $F_1 \equiv F_2 \pmod{p}$ if the congruence relation

$$a(F_1; T) \equiv a(F_2; T) \pmod{p}$$

is satisfied for all $0 \leq T \in \text{Sym}_n^*(\mathbb{Z})$.

2.3. Theta series for lattices and matrices. For a positive definite integral lattice \mathcal{L} of rank m , we write the Gram matrix as $S = S_{\mathcal{L}} \in \text{Sym}_m(\mathbb{Z})$. We associate with it the theta series

$$\vartheta_{\mathcal{L}}^{(n)} = \vartheta_S^{(n)}(Z) := \sum_{X \in M_{m,n}(\mathbb{Z})} \exp(\pi \sqrt{-1} \text{tr}(S[X]Z)), \quad Z \in \mathbb{H}_n.$$

In general, this becomes a Siegel modular form for some congruence subgroup of Γ_n . In particular,

$$\vartheta_{\mathcal{L}}^{(n)} = \vartheta_S^{(n)}(Z) \in M_{\frac{m}{2}}(\Gamma_n)_{\mathbb{Z}}$$

if \mathcal{L} is a positive definite, even, unimodular lattice of rank m .

For our use below, we now quote the following result, which is a special case of a theorem presented by Böcherer and Nagaoka ([3], Theorem 5).

Theorem 3. ([3], Theorem 5) *Assume that $p \geq 2n + 3$ and $p \equiv 3 \pmod{4}$. Let $S \in \text{Sym}_2(\mathbb{Z})$ be a positive definite binary quadratic form with $\det(2S) = p$. Then, there exists a modular form $G \in M_{\frac{p+1}{2}}(\Gamma_n)_{\mathbb{Z}_{(p)}}$ such that*

$$\vartheta_S^{(n)} \equiv G \pmod{p}.$$

Proof. We apply Theorem 5 of [3], where

$$f = \vartheta_S^{(n)} \in M_1(\Gamma_n, \chi_p)^0, \quad g = G \in M_{1+\frac{p-1}{2}}(\Gamma_n).$$

□

2.4. Theta operator and mod p singular modular form. First, we will define the theta operator, which is a differential operator. For $F = \sum a(F; T) q^T \in M_k(\Gamma_n)$, we associate with it the formal power series

$$\Theta(F) := \sum a(F; T) \cdot \det(T) q^T \in \mathbb{C}[q_{ij}, q_{ij}^{-1}] \llbracket q_{11}, \dots, q_{nn} \rrbracket.$$

This is called *the theta operator* (cf. [2]). For $n = 1$, the classical theta operator was studied by Ramanujan [11]. It should be noted that $\Theta(F)$ is not necessarily of modular form.

Next, we introduce the mod p singular modular form [1]. For a prime number p , a modular form $F = \sum a(F; T) q^T \in M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}$ is called *the mod p singular modular form* with the (nontrivial) maximal p -rank r ($r < n$) if F has the following property:

$$a(F; T) \equiv 0 \pmod{p}$$

for all $T \in \text{Sym}_n^*(\mathbb{Z})$ with $r + 1 \leq \text{rank}(T) \leq n$ and

$$a(F; T) \not\equiv 0 \pmod{p}$$

for some T with $\text{rank}(T) = r$.

If $F = \sum a(F; T)q^T \in M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}$ is a mod p singular modular form, then

$$\Theta(F) \equiv 0 \pmod{p}.$$

Namely, F is an element of the mod p kernel of the theta operator.

In this note, we will show that the theta series associated with some Niemeier lattices are examples of such forms.

2.5. Sturm bound for Siegel modular forms. In this section, we introduce a result of Richter and Raum [9] concerning the so-called Sturm bound. From this result, we can specify a modular form by using mod p .

Theorem 4. [9] *Assume that p is a prime number and $F = \sum a(F; T)q^T$ is a modular form in $M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}$ ($n \geq 2$). If*

$$a(F; T) \equiv 0 \pmod{p}$$

for all $0 \leq T = (t_{ij}) \in \text{Sym}_n^*(\mathbb{Z})$ with

$$t_{ii} \leq \left(\frac{4}{3}\right)^n \frac{k}{16}, \quad (i = 1, \dots, n),$$

then

$$a(F; T) \equiv 0 \pmod{p}$$

for all $0 \leq T \in \text{Sym}_n^*(\mathbb{Z})$.

Corollary 4.1. *Let $F = \sum a(F; T)q^T$ be a modular form in $M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}$. If*

$$a(F; T) \in \mathbb{Z}$$

for all $0 \leq T \in \text{Sym}_n^*(\mathbb{Z})$ with

$$t_{ii} \leq \left(\frac{4}{3}\right)^n \frac{k}{16}, \quad (i = 1, \dots, n),$$

then $F \in M_k(\Gamma_n)_{\mathbb{Z}}$.

2.6. Niemeier lattices. A Niemeier lattice is one of the 24 positive definite, even, unimodular lattices of rank 24, which were classified by H. Niemeier. If \mathcal{L} is a Niemeier lattice, then the corresponding theta series $\vartheta_{\mathcal{L}}^{(n)}$ becomes a Siegel modular form of weight 12:

$$\vartheta_{\mathcal{L}}^{(n)} \in M_{12}(\Gamma_n)_{\mathbb{Z}}.$$

As stated in the Introduction, for the Niemeier lattices, we use the notation $\alpha, \beta, \dots, \omega$, as defined by Conway and Sloane ([4], p. 407, Table 16.1). We write the associated theta series as $\vartheta_{\alpha}^{(n)}, \vartheta_{\beta}^{(n)}, \dots$. One of our main purposes in this note is to study these theta series.

3. DEGREE 2 THETA SERIES FOR NIEMEIER LATTICES

3.1. Igusa's generators. Let $E_k^{(n)}$ be the Eisenstein series of degree n and weight k , normalized as $a(E_k^{(n)}; 0_n) = 1$. It is well known that $E_k^{(n)} \in M_k(\Gamma_n)_{\mathbb{Q}}$.

We set

$$M(\Gamma_2)_{\mathbb{Z}} = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma_2)_{\mathbb{Z}}.$$

Igusa [5] gave a minimal set of generators of the ring $M(\Gamma_2)_{\mathbb{Z}}$ over \mathbb{Z} . The set consists of 15 modular forms:

$$M(\Gamma_2)_{\mathbb{Z}} = \mathbb{Z}[X_4, X_6, X_{10}, X_{12}, Y_{12}, X_{16}, \dots, X_{48}].$$

Here, the subscripts denote the weights, and

$$X_4 = E_4^{(2)}, \quad X_6 = E_6^{(2)}, \quad X_{10} = \chi_{10}, \quad X_{12} = \chi_{12},$$

where χ_k ($k = 10, 12$) is Igusa's cusp form normalized as

$$a\left(\chi_k; \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\right) = 1.$$

There are two modular forms of weight 12. The form Y_{12} has the q -expansion

$$\begin{aligned} Y_{12} = & (q_{11} + q_{22}) - 24(q_{11}^2 + q_{22}^2) \\ & + (q_{12}^{-2} + 116q_{12}^{-1} + 1206 + 116q_{12} + q_{12}^2)q_{11}q_{22} + \dots \end{aligned}$$

and satisfies

$$\Phi(Y_{12}) = \Delta,$$

where Φ is the Siegel operator, and

$$\begin{aligned} \Delta : &= \frac{1}{1728} \left((E_4^{(1)})^3 - (E_6^{(1)})^2 \right) \\ &= q - 24q^2 + 252q^3 - 1472q^4 \dots \in S_{12}(\Gamma_1)_{\mathbb{Z}}. \end{aligned}$$

For use below, we now give the expressions for $X_{12}^{(2)} = X_{12}$ and $Y_{12}^{(2)} = Y_{12}$ using the Eisenstein series:

$$\begin{aligned} X_{12}^{(2)} &= a_1 \cdot (E_4^{(2)})^3 + a_2 \cdot (E_6^{(2)})^2 + a_3 \cdot E_{12}^{(2)}, \\ (2) \quad a_1 &= \frac{131 \cdot 593}{2^{11} \cdot 3^4 \cdot 5^3 \cdot 337}, \quad a_2 = \frac{131 \cdot 593}{2^{10} \cdot 3^6 \cdot 7^2 \cdot 337}, \\ a_3 &= \frac{-131 \cdot 593 \cdot 691}{2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 337}, \\ Y_{12}^{(2)} &= b_1 \cdot (E_4^{(2)})^3 + b_2 \cdot (E_6^{(2)})^2 + b_3 \cdot E_{12}^{(2)}, \\ (3) \quad b_1 &= \frac{41 \cdot 71 \cdot 109}{2^7 \cdot 3^3 \cdot 5^3 \cdot 337}, \quad b_2 = \frac{1759}{2^2 \cdot 3^4 \cdot 7^2 \cdot 337}, \\ b_3 &= \frac{-131 \cdot 593 \cdot 691}{2^7 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 337}. \end{aligned}$$

3.2. Theta series for Niemeier lattices of degree 2. Let \mathcal{L} be a Niemeier lattice. It is known that if the Coxeter number of \mathcal{L} is h , then the q -expansion of $\vartheta_{\mathcal{L}}^{(1)}$ is given as follows:

$$\vartheta_{\mathcal{L}}^{(1)} = 1 + 24h \cdot q + \dots$$

Since $\vartheta_{\mathcal{L}}^{(1)} \in \mathbb{Z}[E_4^{(1)}, \Delta]$, the form $\vartheta_{\mathcal{L}}^{(1)}$ can be expressed as

$$(4) \quad \vartheta_{\mathcal{L}}^{(1)} = (E_4^{(1)})^3 + (24h - 720)\Delta.$$

This identity is a starting point for our study.

Theorem 5. *Let \mathcal{L} be a Niemeier lattice with Coxeter number h . Then we have*

$$(5) \quad \vartheta_{\mathcal{L}}^{(2)} = (E_4^{(2)})^3 + (24h - 720)Y_{12}^{(2)} + (48h^2 - 2800h + 43200)X_{12}^{(2)},$$

where $X_{12}^{(2)}, Y_{12}^{(2)}$ are Igusa's generators, which were introduced in 3.1.

Proof. It follows from

$$\Phi(\vartheta_{\mathcal{L}}^{(2)}) = \vartheta_{\mathcal{L}}^{(1)}, \quad \Phi(E_4^{(2)}) = E_4^{(1)}, \quad \Phi(Y_{12}^{(2)}) = \Delta$$

that $\vartheta_{\mathcal{L}}^{(2)}$ has the following expression:

$$(6) \quad \vartheta_{\mathcal{L}}^{(2)} = (E_4^{(2)})^3 + (24h - 720)Y_{12}^{(2)} + c_1 \cdot X_{12},$$

where $c_i = c_i(h)$ is a constant. We shall determine the value c_1 .

We consider the diagonal restriction of both sides of (6). We have

$$\begin{aligned} \vartheta_{\mathcal{L}}^{(2)} \left(\begin{pmatrix} z_{11} & 0 \\ 0 & z_{22} \end{pmatrix} \right) &= \vartheta_{\mathcal{L}}^{(1)}(z_{11}) \cdot \vartheta_{\mathcal{L}}^{(1)}(z_{22}) \\ &= 1 + 24h \cdot q_{11} + 24h \cdot q_{22} + (24h)^2 \cdot q_{11}q_{22} + \cdots, \\ (E_4^{(2)})^3 \left(\begin{pmatrix} z_{11} & 0 \\ 0 & z_{22} \end{pmatrix} \right) &= (E_4^{(1)}(z_{11}) \cdot E_4^{(1)}(z_{22}))^3 \\ &= 1 + 720 \cdot q_{11} + 720 \cdot q_{22} + 720^2 \cdot q_{11}q_{22} + \cdots, \\ Y_{12}^{(2)} \left(\begin{pmatrix} z_{11} & 0 \\ 0 & z_{22} \end{pmatrix} \right) &= E_4^{(1)}(z_{11})\Delta(z_{22}) + E_4^{(1)}(z_{22})\Delta(z_{11}) \\ &= q_{11} + q_{22} + 1440 \cdot q_{11}q_{22} + \cdots, \\ X_{12}^{(2)} \left(\begin{pmatrix} z_{11} & 0 \\ 0 & z_{22} \end{pmatrix} \right) &= 2^2 \cdot 3\Delta(z_{11})\Delta(z_{22}) \\ &= 12 \cdot q_{11}q_{22} + \cdots, \end{aligned}$$

where $q_{ii} = \exp(2\pi\sqrt{-1}z_{ii})$. Comparing the coefficients of $q_{11}q_{22}$, we obtain the identity

$$(24h)^2 = 720^2 + 1440(24h - 720) + 12 \cdot c_1.$$

This implies

$$c_1 = 48h^2 - 2880h + 43200.$$

This completes the proof of Theorem 5. \square

4. DEGREE 3 THETA SERIES FOR NIEMEIER LATTICES

4.1. Siegel modular forms of degree 3 and weight 12. Miyawaki [7] constructed a cusp form $F_{12} \in S_{12}(\Gamma_3)$ by using theta series with a spherical polynomial. In this section, we introduce two modular forms of degree 3 and weight 12. We will then show that they have integral Fourier coefficients.

We set

$$\begin{aligned} X_{12}^{(3)} &:= a_1 \cdot (E_4^{(3)})^3 + a_2 \cdot (E_6^{(3)})^2 + a_3 \cdot E_{12}^{(3)} + \frac{4740}{337}F_{12}, \\ Y_{12}^{(3)} &:= b_1 \cdot (E_4^{(3)})^3 + b_2 \cdot (E_6^{(3)})^2 + b_3 \cdot E_{12}^{(3)} - \frac{356411}{337}F_{12}, \end{aligned}$$

where $F_{12} \in S_{12}(\Gamma_3)_{\mathbb{Z}}$ is Miyawaki's cusp form [7], and a_i and b_i are constants given in (2), and (3) in Section 3.1.

We shall show that they have integral Fourier coefficients. Since F_{12} are in cusp form, we have

$$\Phi(X_{12}^{(3)}) = X_{12}^{(2)}, \quad \Phi(Y_{12}^{(3)}) = Y_{12}^{(2)}.$$

This means that, if $\text{rank}(T) < 3$, then all of the Fourier coefficients $a(X_{12}^{(3)}; T)$ and $a(Y_{12}^{(3)}; T)$ are integral.

In the case that $\text{rank}(T) = 3$, we have the following numerical data for the Fourier coefficients of $X_{12}^{(3)}$, $Y_{12}^{(3)}$, and F_{12} :

T	$a(X_{12}^{(3)}; T)$	$a(Y_{12}^{(3)}; T)$	$a(F_{12}; T)$
$[1, 1, 1; 1, 1, 1]$	1	1	1
$[1, 1, 1; 0, 0, 1]$	84	7674	18
$[1, 1, 1; 0, 0, 0]$	1132	114476	164

Here, we used the abbreviation

$$(7) \quad [a, b, c; d, e, f] := \begin{pmatrix} a & \frac{f}{2} & \frac{e}{2} \\ \frac{f}{2} & b & \frac{d}{2} \\ \frac{e}{2} & \frac{d}{2} & c \end{pmatrix} \in \text{Sym}_3^*(\mathbb{Z}).$$

Proposition 1. *The modular forms $X_{12}^{(3)}$, $Y_{12}^{(3)}$, and F_{12} have integral Fourier coefficients:*

$$X_{12}^{(3)}, Y_{12}^{(3)} \in M_{12}(\Gamma_3)_{\mathbb{Z}}, \quad F_{12} \in S_{12}(\Gamma_3)_{\mathbb{Z}}.$$

Proof. This is true for $F_{12} \in S_{12}(\Gamma_3)_{\mathbb{Z}}$ as a consequence of its definition [7]. We shall show that $X_{12}^{(3)} \in M_{12}(\Gamma_3)_{\mathbb{Z}}$. By Corollary 4.1, it suffices to show that

$$a(X_{12}^{(3)}; T) \in \mathbb{Z}$$

for all $0 \leq T = (t_{ij}) \in \text{Sym}_3^*(\mathbb{Z})$ with $t_{ii} \leq 1$. This can be confirmed from the above table. The same argument can be applied to $Y_{12}^{(3)}$. \square

4.2. Theta series for Niemeier lattices of degree 3. In this section, we show that $\vartheta_{\mathcal{L}}^{(3)}$ (\mathcal{L} : Niemeier lattice) can be expressed as an integral linear combination of $(E_4^{(3)})^3$, $X_{12}^{(3)}$, $Y_{12}^{(3)}$, and F_{12} .

Theorem 6. *Let \mathcal{L} be a Niemeier lattice with Coxeter number h . Then we have*

$$(8) \quad \begin{aligned} \vartheta_{\mathcal{L}}^{(3)} = & (E_4^{(3)})^3 + (24h - 720)Y_{12}^{(3)} + (48h^2 - 2800h + 43200)X_{12}^{(3)} \\ & + (48h^3 - 288h^2 + 3144h - 1131120)F_{12}, \end{aligned}$$

where $E_4^{(3)}$ is the Eisenstein series of degree 3 and weight 4; and $X_{12}^{(3)}$, $Y_{12}^{(3)}$, and F_{12} are modular forms given in Section 4.1.

Proof. We note that

$$\Phi(E_4^{(3)}) = E_4^{(2)}, \quad \Phi(X_{12}^{(3)}) = X_{12}^{(2)}, \quad \Phi(Y_{12}^{(3)}) = Y_{12}^{(2)}.$$

Since $S_{12}(\Gamma_3) = \mathbb{C} \cdot F_{12}$, we can write

$$\begin{aligned} \vartheta_{\mathcal{L}}^{(3)} = & (E_4^{(3)})^3 + (24h - 720)Y_{12}^{(3)} + (48h^2 - 2800h + 43200)X_{12}^{(3)} \\ & + c_2 \cdot F_{12} \end{aligned}$$

for some constant c_2 . By an argument similar to the one used in the proof of Theorem 5, we can express the value c_2 as a polynomial in h . As in the case of

degree 2, we consider the diagonal restrictions:

$$\begin{aligned}
\vartheta_{\mathcal{L}}^{(3)} \left(\begin{pmatrix} z_{11} & 0 & 0 \\ 0 & z_{22} & 0 \\ 0 & 0 & z_{33} \end{pmatrix} \right) &= \cdots + (24h)^3 \cdot q_{11}q_{22}q_{33} + \cdots, \\
\left(E_4^{(3)} \left(\begin{pmatrix} z_{11} & 0 & 0 \\ 0 & z_{22} & 0 \\ 0 & 0 & z_{33} \end{pmatrix} \right) \right)^3 &= \cdots + 373248000 \cdot q_{11}q_{22}q_{33} + \cdots, \\
Y_{12}^{(3)} \left(\begin{pmatrix} z_{11} & 0 & 0 \\ 0 & z_{22} & 0 \\ 0 & 0 & z_{33} \end{pmatrix} \right) &= \cdots + 169632 \cdot q_{11}q_{22}q_{33} + \cdots, \\
X_{12}^{(3)} \left(\begin{pmatrix} z_{11} & 0 & 0 \\ 0 & z_{22} & 0 \\ 0 & 0 & z_{33} \end{pmatrix} \right) &= \cdots + 1728 \cdot q_{11}q_{22}q_{33} + \cdots, \\
F_{12} \left(\begin{pmatrix} z_{11} & 0 & 0 \\ 0 & z_{22} & 0 \\ 0 & 0 & z_{33} \end{pmatrix} \right) &= \cdots + 288 \cdot q_{11}q_{22}q_{33} + \cdots,
\end{aligned}$$

where $q_{ii} := \exp(2\pi\sqrt{-1}z_{ii})$. From these formulas, we obtain

$$\begin{aligned}
(24h)^3 &= 373248000 + 169632 \cdot (24h - 720) \\
&\quad + 1728 \cdot (48h^2 - 2880h + 43200) \\
&\quad + 288 \cdot c_2.
\end{aligned}$$

This implies

$$c_2 = 48h^3 - 288h^2 + 3144h - 1131120.$$

This completes the proof. \square

Remark 1. *The polynomial coefficients in the expression of $\vartheta_{\mathcal{L}}^{(3)}$ are factored as follows:*

$$\begin{aligned}
c_0(h) &:= 24h - 720 = 24(h - 30), \\
(9) \quad c_1(h) &:= 48h^2 - 2880h + 43200 = 48(h - 30)^2, \\
c_2(h) &:= 48h^3 - 288h^2 + 3144h - 1131120 = 24(h - 30)(2h^2 + 48h + 1571).
\end{aligned}$$

All of these coefficients include the factor $h - 30$. If we take $\mathcal{L} = \gamma$ (cf. Conway-Sloane's list [4], p. 407, Table 16.1), then the Coxeter number is just 30. In this case, we have

$$\vartheta_{\gamma}^{(n)} = (E_4^{(n)})^3 \quad (n = 1, 2, 3).$$

This identity is also justified by the fact that

$$\gamma = (E_8)^3,$$

where E_8 is the so-called E_8 lattice.

The following table is obtained from Theorem 6. Here, $h = h_{\mathcal{L}}$ is the Coxeter number of the Niemeier lattice \mathcal{L} .

TABLE 1. Representation of degree three theta series of Niemeier lattices

Name	Components	h	Theta series
α	D_{24}	46	$\vartheta_\alpha^{(3)} = (E_4^{(3)})^3 + 384Y_{12}^{(3)} + 12288X_{12}^{(3)} + 3076224F_{12}$
β	$D_{16}E_8$	30	$\vartheta_\beta^{(3)} = (E_4^{(3)})^3$
γ	E_8^3	30	$\vartheta_\gamma^{(3)} = \vartheta_\beta^{(3)}$
δ	A_{24}	25	$\vartheta_\delta^{(3)} = (E_4^{(3)})^3 - 120Y_{12}^{(3)} + 1200X_{12}^{(3)} - 482520F_{12}$
ϵ	D_{12}^2	22	$\vartheta_\epsilon^{(3)} = (E_4^{(3)})^3 - 192Y_{12}^{(3)} + 3072X_{12}^{(3)} - 690240F_{12}$
ζ	$A_{17}E_7$	18	$\vartheta_\zeta^{(3)} = (E_4^{(3)})^3 - 288Y_{12}^{(3)} + 6912X_{12}^{(3)} - 887904F_{12}$
η	$D_{10}E_7^2$	18	$\vartheta_\eta^{(3)} = \vartheta_\zeta^{(3)}$
θ	$A_{15}D_9$	16	$\vartheta_\theta^{(3)} = (E_4^{(3)})^3 - 336Y_{12}^{(3)} + 9408X_{12}^{(3)} - 957936F_{12}$
ι	D_8^3	14	$\vartheta_\iota^{(3)} = (E_4^{(3)})^3 - 384Y_{12}^{(3)} + 12288X_{12}^{(3)} - 1011840F_{12}$
κ	A_{12}^2	13	$\vartheta_\kappa^{(3)} = (E_4^{(3)})^3 - 408Y_{12}^{(3)} + 13872X_{12}^{(3)} - 1033464F_{12}$
λ	$A_{11}D_7E_6$	12	$\vartheta_\lambda^{(3)} = (E_4^{(3)})^3 - 432Y_{12}^{(3)} + 15552X_{12}^{(3)} - 1051920F_{12}$
μ	E_6^4	12	$\vartheta_\mu^{(3)} = \vartheta_\lambda^{(3)}$
ν	$A_9^2D_6$	10	$\vartheta_\nu^{(3)} = (E_4^{(3)})^3 - 480Y_{12}^{(3)} + 19200X_{12}^{(3)} - 1080480F_{12}$
ξ	D_6^4	10	$\vartheta_\xi^{(3)} = \vartheta_\nu^{(3)}$
o	A_8^3	9	$\vartheta_o^{(3)} = (E_4^{(3)})^3 - 504Y_{12}^{(3)} + 21168X_{12}^{(3)} - 1091160F_{12}$
π	$A_7^2D_5^2$	8	$\vartheta_\pi^{(3)} = (E_4^{(3)})^3 - 528Y_{12}^{(3)} + 23232X_{12}^{(3)} - 1099824F_{12}$
ρ	A_6^4	7	$\vartheta_\rho^{(3)} = (E_4^{(3)})^3 - 552Y_{12}^{(3)} + 25392X_{12}^{(3)} - 1106760F_{12}$
σ	$A_5^4D_4$	6	$\vartheta_\sigma^{(3)} = (E_4^{(3)})^3 - 576Y_{12}^{(3)} + 27648X_{12}^{(3)} - 1112256F_{12}$
τ	D_4^6	6	$\vartheta_\tau^{(3)} = \vartheta_\sigma^{(3)}$
v	A_4^6	5	$\vartheta_v^{(3)} = (E_4^{(3)})^3 - 600Y_{12}^{(3)} + 30000X_{12}^{(3)} - 1116600F_{12}$
ϕ	A_3^8	4	$\vartheta_\phi^{(3)} = (E_4^{(3)})^3 - 624Y_{12}^{(3)} + 32448X_{12}^{(3)} - 1120080F_{12}$
χ	A_2^{12}	3	$\vartheta_\chi^{(3)} = (E_4^{(3)})^3 - 648Y_{12}^{(3)} + 34992X_{12}^{(3)} - 1122984F_{12}$
ψ	A_1^{24}	2	$\vartheta_\psi^{(3)} = (E_4^{(3)})^3 - 672Y_{12}^{(3)} + 37632X_{12}^{(3)} - 1125600F_{12}$
ω	Leech	0	$\vartheta_\omega^{(3)} = (E_4^{(3)})^3 - 720Y_{12}^{(3)} + 43200X_{12}^{(3)} - 1131120F_{12}$

As stated in the Introduction, expression (8) of Theorem 6 is useful for studying the congruences between modular forms. As a straightforward conclusion, we can prove the following result.

Corollary 6.1. *Let \mathcal{L}_i ($i = 1, 2$) be Niemeier lattices with Coxeter number $h_i := h_{\mathcal{L}_i}$. If $h_1 \equiv h_2 \pmod{m}$ for an integer m , then*

$$\vartheta_{\mathcal{L}_1}^{(3)} \equiv \vartheta_{\mathcal{L}_2}^{(3)} \pmod{m}.$$

Example 1. *Since $h_\beta = 30$ and $h_\rho = 7$ (see Table 1), we have*

$$\vartheta_\beta^{(3)} \equiv \vartheta_\rho^{(3)} \pmod{23}.$$

For example,

$$a(\vartheta_\beta^{(3)}; [3, 1, 2]) = 749432632320 \equiv 799943308416 = a(\vartheta_\rho^{(3)}; [3, 1, 2]) \pmod{23}.$$

Here we used the following abbreviation: For $\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \in \text{Sym}_2^*(\mathbb{Z})$, we set

$$(10) \quad [a, b, c] := \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \in \text{Sym}_2^*(\mathbb{Z}).$$

Corollary 6.2. *Let $\{\mathcal{L}_i \ (i = 1, 2, 3, 4)\}$ be a set of Niemeier lattices with Coxeter number $h_i = h_{\mathcal{L}_i}$ such that $h_1 < h_2 < h_3 < h_4$. Then, for any Niemeier lattice \mathcal{L} , the Niemeier theta series $\vartheta_{\mathcal{L}}^{(3)}$ has the following expression:*

$$\vartheta_{\mathcal{L}}^{(3)} = \sum_{j=1}^4 \ell_j(h_{\mathcal{L}}) \vartheta_{\mathcal{L}_j}^{(3)},$$

where $\ell_j(x)$ ($j = 1, 2, 3, 4$) are the Lagrange basis polynomials:

$$\ell_j(x) := \prod_{\substack{1 \leq m \leq 4 \\ m \neq j}} \frac{x - x_m}{x_j - x_m}.$$

Proof. We recall expression (8) of Theorem 6, and we solve the system of equations

$$(11) \quad \vartheta_{\mathcal{L}_i}^{(3)} = (E_4^{(3)})^3 + c_0(h_i)Y_{12}^{(3)} + c_1(h_i)X_{12}^{(3)} + c_2(h_i)F_{12}, \quad (i = 1, 2, 3, 4),$$

with respect to $(E_4^{(3)})^3$, $Y_{12}^{(3)}$, $X_{12}^{(3)}$, and F_{12} . Here, $c_j(h)$ is the polynomial defined in (9) of Remark 1. Since

$$\begin{vmatrix} 1 & c_0(h_1) & c_1(h_1) & c_2(h_1) \\ 1 & c_0(h_2) & c_1(h_2) & c_2(h_2) \\ 1 & c_0(h_3) & c_1(h_3) & c_2(h_3) \\ 1 & c_0(h_4) & c_1(h_4) & c_2(h_4) \end{vmatrix} = 55296 \cdot \Delta(h_1, h_2, h_3, h_4) \neq 0, \quad (\Delta : \text{the differente}),$$

the equations (11) are solvable. Again considering expression (8), we conclude that $\vartheta_{\mathcal{L}}^{(3)}$ has the following expression:

$$\vartheta_{\mathcal{L}}^{(3)} = \sum_{j=1}^4 f_j(h_{\mathcal{L}}) \vartheta_{\mathcal{L}_j}^{(3)},$$

for some $f_j(x) \in \mathbb{Q}(h_1, h_2, h_3, h_4)[x]$. A direct calculation shows that

$$f_j(x) = \prod_{\substack{1 \leq m \leq 4 \\ m \neq j}} \frac{x - x_m}{x_j - x_m} = \ell_j(x).$$

This completes the proof of Corollary 6.2. \square

5. CONGRUENCE PROPERTIES OF THETA SERIES FOR NIEMEIER LATTICES

5.1. Congruence relation between theta series. We will now prove some congruence relations satisfied by the Niemeier theta series. For this, we need information about the Fourier coefficients of the generators $(E_4^{(3)})^3$, $Y_{12}^{(3)}$, $X_{12}^{(3)}$, and F_{12} .

The results in [8] and [6] can help us to calculate the Fourier coefficients of the Eisenstein series $E_k^{(3)}$.

By combining the Fourier coefficients of Miyawaki's cusp form F_{12} (cf. [7]), we obtain numerical examples of the Fourier coefficients of $\vartheta_{\mathcal{L}}^{(3)}$ (\mathcal{L} : Niemeier lattice). We have the following result:

Theorem 7. *The following congruence relations hold:*

$$(12) \quad \begin{aligned} \vartheta_{\alpha}^{(3)} &\equiv \vartheta_{\omega}^{(3)} \equiv \vartheta_{[4,2,6]}^{(3)} \pmod{23}, \\ \vartheta_{\delta}^{(3)} &\equiv \vartheta_{\psi}^{(3)} \equiv \vartheta_{[2,2,12]}^{(3)} \pmod{23}, \end{aligned}$$

where α , δ , ψ , and ω are Niemeier lattices listed in Table 1.

Proof. We prove the first congruence relation. Since $\det([4, 2, 6]) = 23 \equiv 3 \pmod{4}$, we can apply Theorem 3 to $\vartheta_{[4,2,6]}^{(3)}$. As a consequence, there is a modular form $G_1 \in M_{12}(\Gamma_3)_{\mathbb{Z}_{(23)}}$ such that

$$\vartheta_{[4,2,6]}^{(3)} \equiv G_1 \pmod{23}.$$

We obtain the following tables:

T	$a(\vartheta_{[4,2,6]}^{(2)}; T)$	$a(\vartheta_{\alpha}^{(2)}; T)$	$a(\vartheta_{\omega}^{(2)}; T)$
$[0, 0, 0]$	1	1	1
$[1, 0, 0]$	0	1104	0
$[1, 1, 1]$	0	97152	0
$[1, 0, 1]$	0	1022304	0

T	$a(\vartheta_{[4,2,6]}^{(3)}; T)$	$a(\vartheta_{\alpha}^{(3)}; T)$	$a(\vartheta_{\omega}^{(3)}; T)$
$[1, 1, 1; 1, 1, 1]$	0	4177536	0
$[1, 1, 1; 0, 0, 1]$	0	81607680	0
$[1, 1, 1; 0, 0, 0]$	0	781393536	0

Here, we used the abbreviation $[a, b, c; d, e, f]$ introduced in Section 4.1, (7). From the information in the above tables, we can show that

$$a(\vartheta_{[4,2,6]}^{(3)}; T) \equiv a(G_1; T) \equiv a(\vartheta_{\alpha}^{(3)}; T) \equiv a(\vartheta_{\omega}^{(3)}; T) \pmod{23}$$

for all $T = (t_{ij}) \in \text{Sym}_3^*(\mathbb{Z})$ with $t_{ii} \leq 1$. By using Theorem 4, we obtain

$$\vartheta_{[4,2,6]}^{(3)} \equiv \vartheta_{\alpha}^{(3)} \equiv \vartheta_{\omega}^{(3)} \pmod{23}.$$

The proof of the second congruence relation proceeds in a similar manner. There is a modular form $G_2 \in M_{12}(\Gamma_3)_{\mathbb{Z}_{(23)}}$ such that

$$\vartheta_{[2,2,12]}^{(3)} \equiv G_2 \pmod{23}.$$

In this case, we obtain the following tables:

T	$a(\vartheta_{[2,2,12]}^{(2)}; T)$	$a(\vartheta_{\delta}^{(2)}; T)$	$a(\vartheta_{\psi}^{(2)}; T)$
$[0, 0, 0]$	1	1	1
$[1, 0, 0]$	2	600	48
$[1, 1, 1]$	0	27600	0
$[1, 0, 1]$	0	303600	2208

T	$a(\vartheta_{[2,2,12]}^{(3)}; T)$	$a(\vartheta_\delta^{(3)}; T)$	$a(\vartheta_\psi^{(3)}; T)$
$[1, 1, 1; 1, 1, 1]$	0	607200	0
$[1, 1, 1; 0, 0, 1]$	0	12751200	0
$[1, 1, 1; 0, 0, 0]$	0	127512000	97152

Again, by Theorem 4, we obtain

$$\vartheta_{[2,2,12]}^{(3)} \equiv G_2 \equiv \vartheta_\delta^{(3)} \equiv \vartheta_\psi^{(3)} \pmod{23}.$$

This completes the proof. \square

Remark 2. *The congruence relations*

$$\vartheta_\alpha^{(3)} \equiv \vartheta_\omega^{(3)} \pmod{23} \text{ and } \vartheta_\delta^{(3)} \equiv \vartheta_\psi^{(3)} \pmod{23}$$

can be also proved by Corollary 6.1.

5.2. Theta operator on theta series and the mod p singular form. In the previous section, we saw some congruence relations arising from theta series. Such relations can be reformulated by the terminology of the theta operator and the mod p singular forms that were introduced in Section 2.4.

Theorem 8. (1) *The following congruence relations hold:*

$$\Theta(\vartheta_\alpha^{(2)}) \equiv \Theta(\vartheta_\delta^{(2)}) \equiv \Theta(\vartheta_\psi^{(2)}) \equiv \Theta(\vartheta_\omega^{(2)}) \equiv 0 \pmod{23}.$$

(2) *The theta series $\vartheta_\alpha^{(3)}$, $\vartheta_\delta^{(3)}$, $\vartheta_\psi^{(3)}$, and $\vartheta_\omega^{(3)}$ are mod 23 singular modular forms with the nontrivial maximal 23-rank 2.*

Proof. Statement (1) is a consequence of Theorem 7. To prove statement (2), we must show that

$$a(\vartheta_\psi^{(3)}; T) \not\equiv 0 \pmod{23}$$

for some T with $\text{rank}(T) = 2$. This comes from the following:

$$\begin{aligned} a(\vartheta_\alpha^{(2)}; [2, 1, 3]) &\equiv a(\vartheta_\delta^{(2)}; [1, 1, 6]) \\ &\equiv a(\vartheta_\psi^{(2)}; [1, 1, 6]) \equiv a(\vartheta_\omega^{(2)}; [2, 1, 3]) \\ &\equiv 2 \pmod{23}. \end{aligned}$$

\square

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